The Geometry of Piles of Salt

Math Teacher's Circle 9-6-2016
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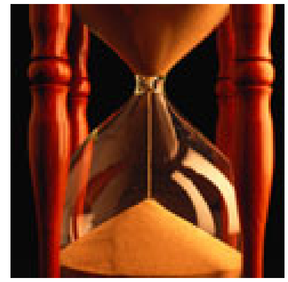

Introduction

A few summers ago I attended a math workshop where participants were encouraged to share some of their teaching ideas. A professor from Brazil shared some ideas that forever changed the way I look at the world. I'll never forget the exhilaration I experienced as I predicted in my mind what would happen next as he performed his experiments. French novelist and philosopher Marcel Proust (1871-1922) penned these sentiments which captured the way I felt: "The real voyage of discovery lies not in finding new lands, but in seeing with new eyes."

As I have investigated his ideas, and come up with proofs of each case, I have had to research many topics in order to tie loose ends together. I have experienced much satisfaction in this process. Although I will not supply the proofs in this article, I would encourage those interested to prove the propositions put forth in each experiment.

A Few Preliminaries

When granular material is poured on a surface from a fixed source, it begins to form a conical pile. The angle that the surface of the pile of granular material makes with horizontal is referred to as the angle of repose. The *angle of repose* is the minimum angle at which this granular material can no longer support itself, and will flow under the influence of gravity. (The term "granular" covers a wide range, since even large boulders that accumulate at the foot of a mountain have an angle of repose, and a rockslide or avalanche occurs if this angle is exceeded.)

The steepness of the angle of repose is affected by such properties as the size and angularity of the grains, density of the grains, cohesion between the grains (due to electrostatic energy, magnetism, water film, etc.), substrate roughness, shear-stress, and gravity. Ordinary table salt has an angle of repose of about 32°.

The Method of Investigation

In order to perform these experiments with students, you will need several boxes of table salt which you can sometimes find on sale 3 for a dollar, and several paper cups. I like to have2 sheets of poster paper or butcher paper on the table to catch salt and then lift it back up and funnel it into the paper cups to reuse on the subsequent trials.

You will need to cut out of cardboard or other stiff material various polygons and other circular shapes as described later on in this article. I like to keep the shapes fairly small so as not to need too much salt to cover them completely. There are several layers of mathematical sophistication that can be pointed out in each experiment, depending on the background of the students.

First, after placing the two sheets of poster paper on the table, one on top of the other, I place one or more paper cups on the poster paper and then suspend one of the polygons on the paper cups. I begin pouring the salt on the polygon until the shape is completely filled with salt and it begins to slide off the edges of the polygon into the cups and onto the paper. As you are beginning to pour, and throughout the experiment, you should be asking students to make conjectures and predict what will happen. As their predictions are either verified or contradicted, you should try to have students explain what is happening and guide them into discovering as much of the mathematics as their background allows.

Triangle Shape

As the salt begins to fill the fill the interior of the triangle and then slide off the edges, ridgelines begin to form. These ridgelines appear to all meet at a point, forming a triangular pyramid (see figure 1). It is interesting to try to guess what this point is where the ridgelines meet. As is the case with all of these experiments, an observer looking straight down on the salt pile could imagine projecting the ridgelines and points directly down to the base figure. Then the ridgelines and points can be talked about in a two-dimensional environment.

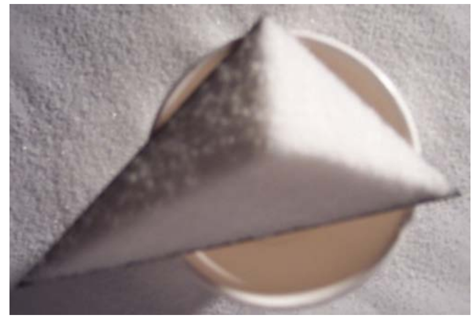

Figure 1

The most common points of concurrency discussed in a geometry class are the centroid, incenter, circumcenter, and orthocenter. The centroid is the point of concurrency of the medians, the segments connecting the midpoint of a side with the opposite vertex. The incenter is the point of concurrency of the angle bisectors. The circumcenter is where the perpendicular bisectors of the sides concur. And the altitudes of a triangle concur at the orthocenter.

Look at the physical model of what is happening to verify the geometry. As the salt slides off the edges of the triangle, the ridgelines stabilize at a location that is equidistant from the edges of the triangle. In a triangle the line that is equidistant from two sides of a triangle is the bisector of the angle formed by those two sides. Therefore, each ridge line of the pyramid, when projected down to the base triangle, is an angle bisector. The angle bisectors meet at the incenter, the point that is equidistant from the sides of the triangle. So the peak of the pyramid is directly above the incenter of the triangular base.

Quadrilateral Shape

When salt is poured on an arbitrary quadrilateral shape, ridgelines are also formed, but they do not necessarily all meet at a single point (see figure 2). As with the triangle, we can see that the ridgelines of the quadrilateral originating from each vertex are angle bisector of the angles at those vertices. The ridgelines originating from the two closest vertices meet at a point. The ridgelines originating from the next pair of closest vertices also meet at a point. These two points are then connected by another ridgeline. After some thought, it can be determined that this new ridgeline is equidistant from the two closest sides of the quadrilateral, and is therefore the angle bisector of the angle formed by extending these two sides until they meet. It is an interesting investigation to determine which types of quadrilaterals have ridgelines that all concur at a single point.

Figure 2

Circular Shapes

If a circle is cut out of cardboard, and then another circle is cut out of the interior of the circle, interesting investigations can be performed. The inner circle can be concentric with the outer circle, or offset towards the edge. As salt is poured on the circular shape and slides off the edges of the outer and inner circle, a ridgeline is formed that is equidistant from both edges (see figure 3). It is interesting to prove from the geometry of this model that the ridgeline is actually an ellipse.

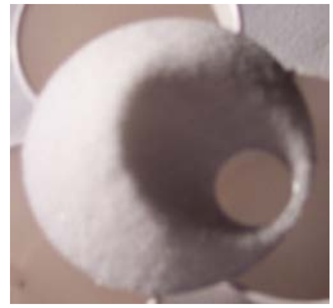

Figure 3

Both of the other conic sections, the parabola and the hyperbola, can also be modeled with salt piles. I have attempted to take photographs where the lighting and shading emphasize the ridgelines. The photographs are imported into Geometer's Sketchpad where a construction can be manipulated to fit on top of the photographs.

The difficulty with modeling the parabola and the hyperbola is that both of these curves are infinite. Figure 4 shows a portion of the parabola, while figure 5 shows a portion of the hyperbola. Because we can't use an infinite amount of salt, secondary ridgelines begin to form where we taper off the salt pile.

Figure 5

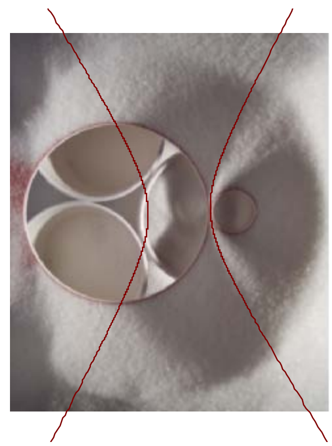

Figure 4

Conclusion

These salt piles provide a rich source of investigation into both simple and deep geometric relationships. While this article is not long enough to contain the proofs of the proposed phenomena, nor a more detailed explanation of my procedures, I would encourage interested readers to investigate these relationships and provide the proofs for themselves. I would be happy to be a resource to anyone wanting to know more on these topics.